Electrostatics

Coulomb's Law

In 1785, Coulomb established the fundamental law of ***electric force*** between two stationary, charged particles. Experiments show that an electric force has the following properties:

(1) The force is *inversely proportional* to the square of separation, r^2, between the two charged particles.

$$F \propto \frac{1}{r^2}$$

(2) The force is *proportional* to the product of charge q_1 and the charge q_2 on the particles.

$$F \propto q_1 q_2$$

(3) The force is *attractive* if the charges are of opposite sign and *repulsive* if the charges have the same sign.

We can conclude that

$$F \propto \frac{q_1 q_2}{r^2}$$

$$\therefore F = K\frac{q_1 q_2}{r^2}$$

where K is the coulomb constant = 9×10^9 N.m^2/C^2.

The above equation is called ***Coulomb's law***, which is used to calculate the force between electric charges. In that equation F is measured in Newton (N), q is measured in unit of coulomb (C) and r in meter (m).

The constant K can be written as

$$K = \frac{1}{4\pi\varepsilon_o}$$

where ε_o is known as the *Permittivity constant of free space.*

$$\varepsilon_o = 8.85 \times 10^{-12}\ \mathrm{C^2/N.m^2}$$

$$K = \frac{1}{4\pi\varepsilon_o} = \frac{1}{4\pi \times 8.85 \times 10^{-12}} = 9 \times 10^9 N.m^2/C^2$$

Electric Field

Definition of the electric field

The electric field vector $\vec{E}$ at a point in space is defined as the electric force $\vec{F}$ acting on a positive test charge placed at that point divided by the magnitude of the test charge q_o

$$\vec{E} = \frac{\vec{F}}{q_o}$$

The electric field has a unit of N/C

لاحظ هنا أن المجال الكهربي $\vec{E}$ هو مجال خارجي وليس المجال الناشئ من الشحنة q_o كما هو موضح في الشكل . وقد يكون هناك مجال كهربي عند أية نقطة في الفراغ بوجود أو عدم وجود الشحنة q_o ولكن وضع الشحنة q_o عند أية نقطة في الفراغ هو وسيلة لحساب المجال الكهربي من خلال القوى الكهربية المؤثرة عليها.

The direction of $\vec{E}$

If Q is +ve the electric field at point p in space is radially outward from Q as shown in figure (a).

If Q is -ve the electric field at point p in space is radially inward toward Q as shown in figure (b).

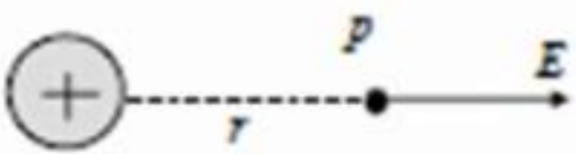


Figure (a)

Figure (b)

يكون اتجاه المجال عند نقطة ما لشحنة موجبة في اتجاه الخروج من النقطة كما في الشكل (a) ، ويكون اتجاه المجال عند نقطة ما لشحنة سالبة في اتجاه الدخول من النقطة إلى الشحنة كما في الشكل (b).

Calculating $\vec{E}$ due to a charged particle

Consider Fig. (a) above, the magnitude of force acting on q_o is given by Coulomb's law

$$F = \frac{1}{4\pi\varepsilon_o}\frac{Qq_o}{r^2}$$

$$E = \frac{F}{q_o}$$

$$E = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$$

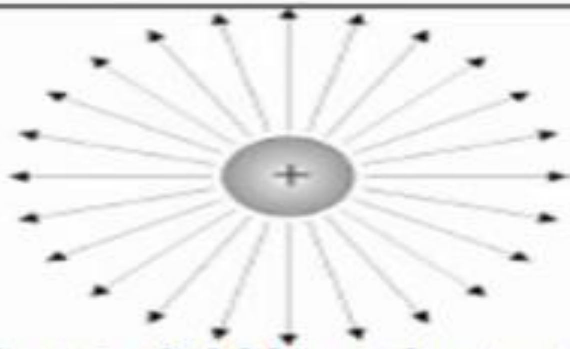
Electric field lines

The electric lines are a convenient way to visualize the electric filed patterns. The relation between the electric field lines and the electric field vector is this:

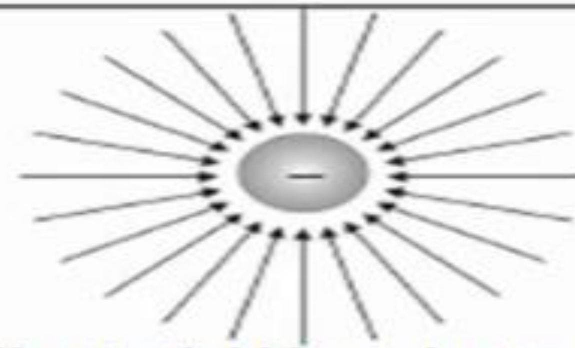
(1) The tangent to a line of force at any point gives the direction of $\vec{E}$ at that point.

(2) The lines of force are drawn so that the number of lines per unit cross-sectional area is proportional to the magnitude of E.

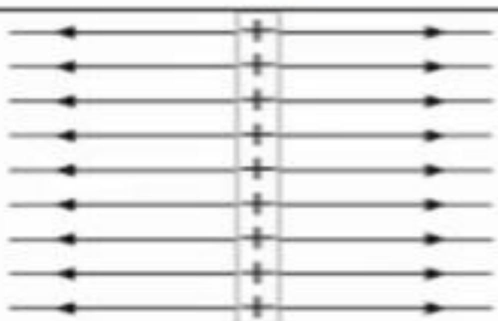
Some examples of electric line of force



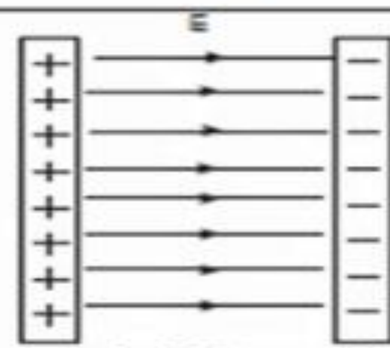
Electric field lines due to +ve charge



Electric field lines due to -ve charge



Electric field lines due to +ve line charge



Electric field lines due two surface charge

Motion of charge particles in a uniform electric field

If we are given a field $\vec{E}$, what forces will act on a charge placed in it?

We start with special case of a point charge in uniform electric field $\vec{E}$. The electric field will exert a force on a charged particle is given by

$$F = qE$$

The force will produce acceleration

$$a = F/m$$

where m is the mass of the particle. Then we can write

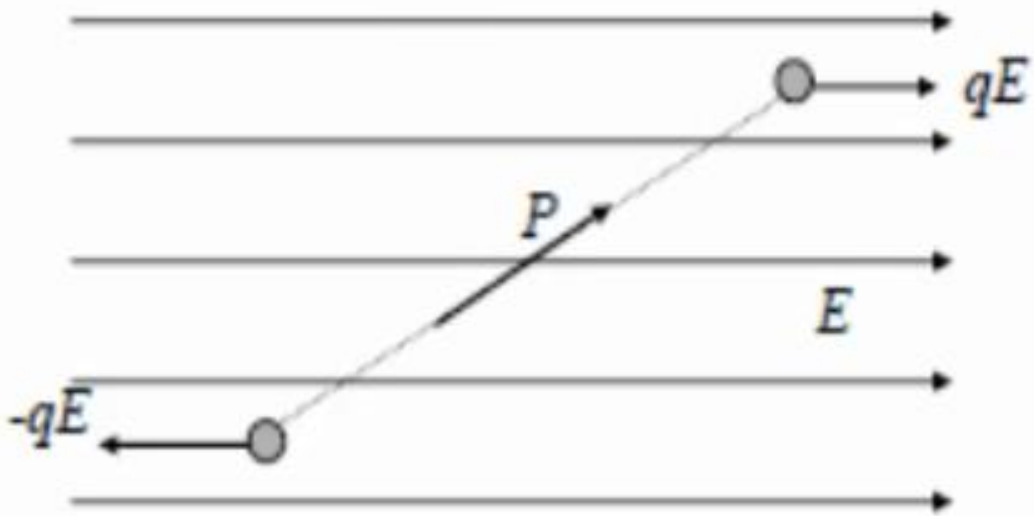
$$F = qE = ma$$

The acceleration of the particle is therefore given by

$$a = qE/m$$

If the charge is positive, the acceleration will be in the direction of the electric field. If the charge is negative, the acceleration will be in the direction opposite the electric field.

If an electric dipole placed in an external electric field E as shown in figure then a torque will act to align it with the direction of the field.



$$\vec{\tau} = \vec{P} \times \vec{E}$$

$$\tau = P\,E \sin\theta$$

where P is the electric dipole momentum, θ the angle between P and E

The Electric Flux due to an Electric Field

We have already shown how electric field can be described by lines of force. A line of force is an imaginary line drawn in such a way that its direction at any point is the same as the direction of the field at that point. Field lines never intersect, since only one line can pass through a single point.

The Electric flux (Φ) is a measure of the number of electric field lines penetrating some surface of area A.

Case one:

The electric flux for a plan surface perpendicular to a uniform electric field (figure 1)

To calculate the electric flux we recall that the number of lines per unit area is proportional to the magnitude of the electric field. Therefore, the number of lines penetrating the surface of area A is proportional to the product EA. The product of the electric filed E and the surface area A perpendicular to the field is called the electric flux Φ.

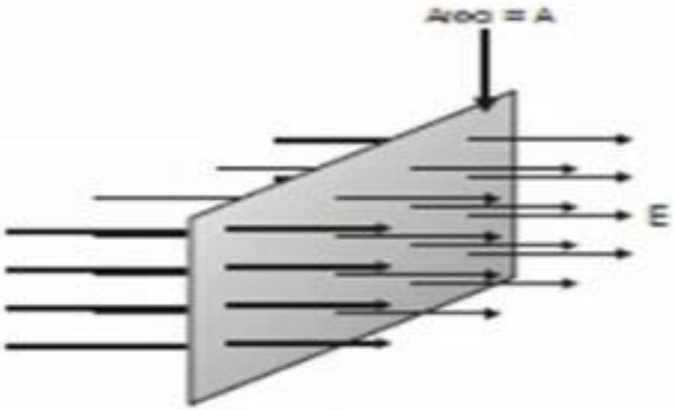


Figure 1

$$\Phi = E.\vec{A} \qquad (4.1)$$

The electric flux Φ has a unit of $N.m^2/C$.

Case Two

The electric flux for a plan surface make an angle θ to a uniform electric field (figure 2)

Note that the number of lines that cross-area is equal to the number that cross the projected area $A^{`}$, which is perpendicular to the field. From the figure we see that the two area are related by $A^{`}=A\cos\theta$. The flux is given by:

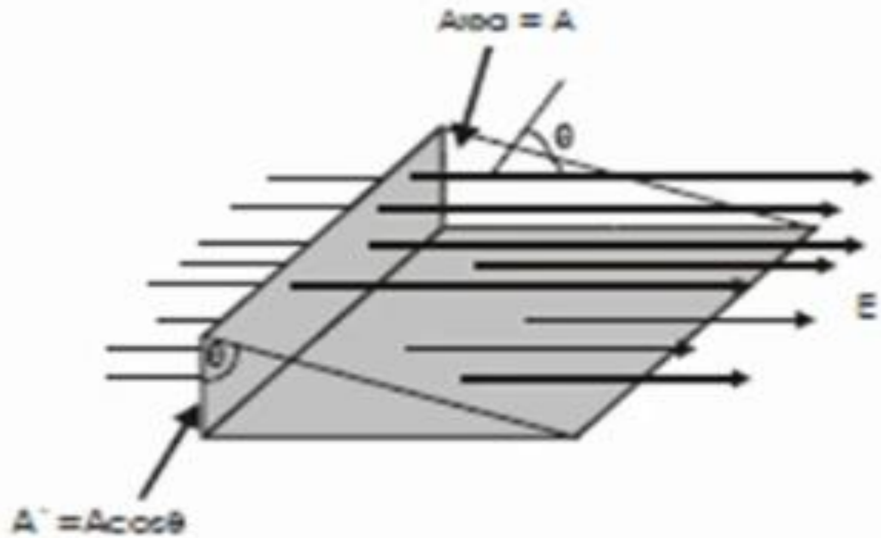


Figure 2

$$\Phi = \vec{E}.\vec{A}' = E\,A\,\cos\theta$$

$$\Phi = \vec{E}.\vec{A}$$

Where θ is the angle between the electric field E and the normal to the surface $\vec{A}$.

إذاً يكون الفيض ذا قيمة عظمى عندما يكون السطح عمودياً على المجال أي 0 = θ ويكون ذا قيمة صغرى عندما يكون السطح موازياً للمجال أي عندما 90 = θ. لاحظ هنا أن المتجه $\vec{A}$ هو متجه المساحة وهو عمودي دائما على المساحة وطوله يعبر عن مقدار المساحة.

Case Three

In general the electric field is nonuniform over the surface (figure .3)

The flux is calculated by integrating the normal component of the field over the surface in question.

$$\Phi = \oint \vec{E} . \vec{A}$$

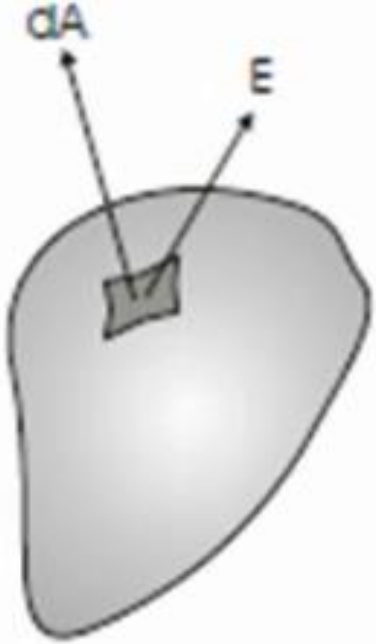


Figure 3

The ***net flux*** through the surface is proportional to the ***net number of lines*** penetrating the surface

والمقصود بـ . net number of lines أي عدد الخطوط الخارجة من السطح (إذا كانت الشحنة موجبة) - عدد الخطوط الداخلة إلى السطح (إذا كانت الشحنة سالبة).

Gauss's law

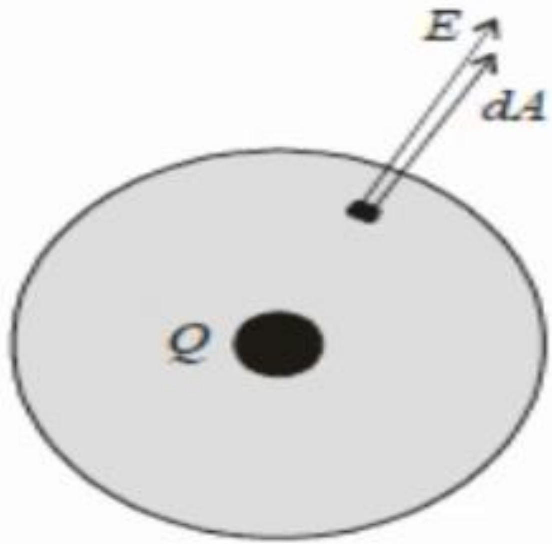
Gauss's law states that the net electric flux through any closed gaussian surface is equal to the net electric charge inside the surface divided by the permittivity.

$$\oint \vec{E}.d\vec{A} = \frac{q_{in}}{\varepsilon_o} \qquad \text{Gauss's law}$$

where q_{in} is the total charge inside the gaussian surface.

Gauss's law and Coulomb's law

We can deduce Coulomb's law from Gauss's law by assuming a point charge q, to find the electric field at point or points a distance r from the charge we imagine a spherical gaussian surface of radius r and the charge q at its center as shown in figure



$$\oint \vec{E} . d\vec{A} = \frac{q_{in}}{\varepsilon_o}$$

$$\oint E \cos 0 dA = \frac{q_{in}}{\varepsilon_o}$$

Because E is constant for all points on the sphere, it can be factored from the inside of the integral sign, then

$$E\oint dA = \frac{q_{in}}{\varepsilon_o} \quad \Rightarrow \quad EA = \frac{q_{in}}{\varepsilon_o} \quad \Rightarrow \quad E(4\pi r^2) = \frac{q_{in}}{\varepsilon_o}$$

$$\therefore E = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}$$

Now put a second point charge q_o at the point, which E is calculated. The magnitude of the electric force that acts on it $F = Eq_o$

$$\therefore F = \frac{1}{4\pi\varepsilon_o}\frac{qq_o}{r^2}$$

ELECTRIC POTENTIAL

Suppose we wish to move a point charge Q from point A to point B in an electric field $\mathbf{E}$ as shown in Figure From Coulomb's law, the force on Q is $\mathbf{F} = Q\mathbf{E}$ so that the *work done* in displacing the charge by $d\mathbf{l}$ is

$$dW = -\mathbf{F} \cdot d\mathbf{l} = -Q\mathbf{E} \cdot d\mathbf{l}$$

The negative sign indicates that the work is being done by an external agent.

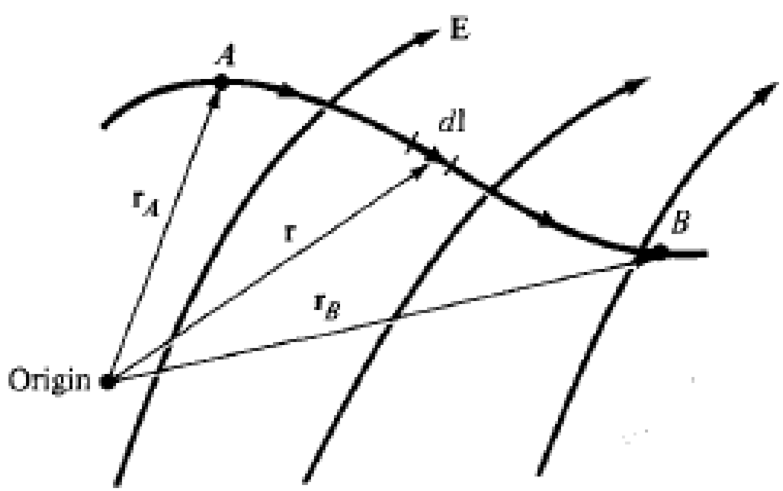


Figure Displacement of point charge Q in an electrostatic field $\mathbf{E}$.

Thus the total work done, or the potential energy required, in moving Q from A to B is

$$\boxed{W = -Q \int_A^B \mathbf{E} \cdot d\mathbf{l}}$$

Dividing W by Q in eq. gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as the *potential difference* between points A and B. Thus

$$\boxed{V_{AB} = \frac{W}{Q} = -\int_A^B \mathbf{E} \cdot d\mathbf{l}}$$

Note that

1. In determining V_{AB}, A is the initial point while B is the final point.
2. If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B; this implies that the work is being done by the field. However, if V_{AB} is positive, there is a gain in potential energy in the movement; an external agent performs the work.
3. V_{AB} is independent of the path taken
4. V_{AB} is measured in joules per coulomb, commonly referred to as volts (V).

As an example, if the **E** field in Figure is due to a point charge Q located at the origin, then

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r \qquad (1)$$

so eq. $V_{AB} = \frac{W}{Q} = -\int_A^B \mathbf{E} \cdot d\mathbf{l}$ becomes

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r \cdot dr\,\mathbf{a}_r \qquad (2)$$

$$= \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

or

$$V_{AB} = V_B - V_A$$

where V_B and V_A are the *potentials* (or *absolute potentials*) at B and A, respectively. Thus the potential difference V_{AB} may be regarded as the potential at B with reference to A. In problems involving point charges, it is customary to choose infinity as reference; that is, we assume the potential at infinity is zero. Thus if $V_A = 0$ as $r_A \to \infty$ in eq. (2), the potential at any point ($r_B \to r$) due to a point charge Q located at the origin is

$$\boxed{V = \frac{Q}{4\pi\varepsilon_o r}}$$

The **potential** at any point is the potential difference between that point and a chosen point at which the potential is zero.

EXAMPLE

Two point charges $-4\ \mu C$ and $5\ \mu C$ are located at $(2, -1, 3)$ and $(0, 4, -2)$, respectively. Find the potential at $(1, 0, 1)$ assuming zero potential at infinity.

Solution:

Let

$$Q_1 = -4\ \mu C, \qquad Q_2 = 5\ \mu C$$

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|} + C_o$$

If $V(\infty) = 0$, $C_o = 0$,

$$|\mathbf{r} - \mathbf{r}_1| = |(1, 0, 1) - (2, -1, 3)| = |(-1, 1, -2)| = \sqrt{6}$$

$$|\mathbf{r} - \mathbf{r}_2| = |(1, 0, 1) - (0, 4, -2)| = |(1, -4, 3)| = \sqrt{26}$$

Hence

$$V(1, 0, 1) = \frac{10^{-6}}{4\pi \times \dfrac{10^{-9}}{36\pi}}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]$$

$$= 9 \times 10^3\,(-1.633 + 0.9806)$$

$$= -5.872\ \text{kV}$$

RELATIONSHIP BETWEEN E AND *V*— MAXWELL'S EQUATION

As shown in the previous section, the potential difference between points A and B is independent of the path taken. Hence,

$$V_{BA} = -V_{AB}$$

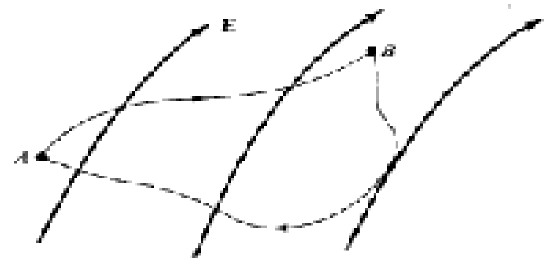


Figure Conservative nature of an electrostatic field

that is, $V_{BA} + V_{AB} = \oint \mathbf{E} \cdot d\mathbf{l} = 0$

or

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = 0} \qquad (1)$$

This shows that the line integral of $\mathbf{E}$ along a closed path as shown in Figure must be zero. Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field. Applying Stokes's theorem to eq. gives

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0$$

or

$$\boxed{\nabla \times \mathbf{E} = 0} \qquad (2)$$

Any vector field that satisfies eq. (1) or (2) is said to be conservative,

Thus an electrostatic field is a conservative field. Equation (1) or (2) is referred to as *Maxwell's equation* (the second Maxwell's equation to be derived) for static electric fields. Equation (1) is the integral form, and eq. (2) is the differential form; they both depict the conservative nature of an electrostatic field.

From the way we defined potential, $V = -\int \mathbf{E} \cdot d\mathbf{l}$, it follows that

$$dV = -\mathbf{E} \cdot d\mathbf{l} = -E_x\,dx - E_y\,dy - E_z\,dz$$

But

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Comparing the two expressions for dV, we obtain

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad E_z = -\frac{\partial V}{\partial z}$$

Thus:

$$\boxed{\mathbf{E} = -\nabla V}$$

that is, the electric field intensity is the gradient of V. The negative sign shows that the direction of $\mathbf{E}$ is opposite to the direction in which V increases; $\mathbf{E}$ is directed from higher to lower levels of V.